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Thermodynamically consistent relations involving plasticity, internal energy and thermal effects

Q1

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Experimental data associated with plastic deformations indicate that the temperature is less than that predicted from dissipation based on plastic work. To obtain reasonable correlation between theoretical and experimental results, the plastic work is often multiplied by a constant β. This paper provides an alternative thermodynamic framework in which it is proposed that there is an additional internal energy associated with dislocation pile-up or increase in dislocation density. The form of this internal energy follows from experimental data that relates flow stress to dislocation density and to equivalent plastic strain. The result is that β is not a constant but a derived function. Representative results for β and temperature as functions of effective plastic strain are provided for both an uncoupled and a coupled thermoplastic theory. In addition to providing features that are believed to be representative of many metals, the formulation can be used as a basis for more advanced theories such as those needed for large deformations and general forms of internal energy.

Q2 **Keywords: Please supply a minimum of three to a maximum of six keywords**

1. Introduction

Experimental observations indicate that many metals display a very specific correlation between temperature and the amount of plastic work performed on the sample. An elementary argument suggests that the increase in the temperature should be proportional to the plastic work but experimental observations always appear to be lower than the prediction. Therefore, it has become common practice to introduce a factor, β, to provide a better correlation between plastic work and predictions of temperature (Taylor & Quinney 1933; Zehnder 1991; Hodowany *et al.* 2000). An extensive survey is provided by Macdougall (2000).

Normally, it is assumed that β is constant. However, Hodowany *et al.* (2000) provide experimental data to demonstrate that such an assumption is not valid. These authors (Rosakis *et al.* 2000) have since provided a systematic, thermodynamical development of a theory in which β is a rationally-derived function.

One contribution of 7 to a Theme Issue ‘Thermodynamics in solids mechanics’.

Here we provide a similar unified framework with a slightly different starting point. The basic premise of this work is that strain hardening is a result of dislocation pile-up or increase in dislocation density. Furthermore, the dislocation density saturates at a certain value beyond which the material behaves as if it is perfectly plastic. An alternative approach is to use effective plastic strain as a history variable but a simple transformation renders these two approaches equivalent. In fact there is a wide range of choices of history variables and history functions that allow for matching theoretical with experimental data. One of the major objectives of the current development is to provide a generic method for developing plasticity models under the assumption that flow stress and temperature measurements are available as experimental data.

First, in §2, we introduce the basic structure in a format that is closely related to existing uncoupled formulations. The internal energy is postulated as a sum of three separable functions involving entropy, elastic strain and internal variables related to plasticity. The form of the evolution functions for plastic strain and dislocation density are left arbitrary and the restriction of the dissipation inequality is used to suggest the development of a yield function. This procedure is somewhat at odds with the more conventional approach of postulating a yield function that depends on conjugate stresses and then deriving evolution functions using associativity. The reason for the current approach is to merely emphasize the point that a restriction such as the choice of a convex yield surface is not necessary based on the dissipation inequality, and that the selection of evolution functions can be based on experimental data. Of course, the two approaches are equivalent under a suitable assumption. The section concludes with the definition of the parameter β that is taken to be a constant in many numerical simulations.

In §3, we make several simplifying assumptions that provide an elementary theory closely relayed to the von Mises formulation that is widely used. In particular, a special form is assumed for the portion of the internal energy associated with dislocation pile-up. The result is that β becomes a derived function that provides basic features often seen with experimental data. The formulation can be easily incorporated with most existing computational algorithms but, more importantly, modifications to the model can be easily made to provide correlations with experimental data. The theory is summarized in terms of dimensionless variables in §4. An additional term is included in the dislocation internal energy and representative results in the form of plots are given to show essential features of the model.

Section 5 provides a coupled theory. The uncoupled formulation is generalized in a rather conventional engineering manner by introducing a thermal strain and by assuming that this strain and the elastic and plastic material parameters depend on (specific) entropy. However, the entropy part of the internal energy is assumed to remain uncoupled. This has the desirable effect of allowing the temperature to be considered as the sum of two terms, an 'entropy' temperature and an additional temperature that is shown to be insignificant. The result is a significant simplification to the coupled theory. Representative results are presented under the assumption that elasticity and plasticity parameters decay exponentially with temperature. Again, illustrative plots are provided.

The most significant aspects of the formulation are summarized in §6.

2. Plasticity

(a) Basic equations for uncoupled thermal effects

To retain the utmost simplicity, deformations will be considered infinitesimal so there is no need to differentiate among the various forms of stress, strain and their rates.

It is quite possible that a material will have been work-hardened, partially annealed, or loaded previously with the result that some initial plasticity exists. Then there is no way of knowing what the initial yield stress should be without an experimental measurement. However, it is known that dislocation pileups can be correlated with initial yield stress and dislocation density, ρ^{d}, can be measured microscopically. Therefore, it might be more meaningful to use dislocation density rather than effective plastic strain as an internal variable with the understanding that the initial value of ρ^{d} may not be zero. Furthermore, with large plastic strain, the dislocation density asymptotically approaches a saturation level, $\rho_{\mathrm{s}}^{\mathrm{d}}$. For convenience we define a dimensionless dislocation parameter, $\tilde{\rho}^{\mathrm{d}}=\rho^{\mathrm{d}}/\rho_{\mathrm{s}}^{\mathrm{d}}$, which will be used in the subsequent development.

Dislocation pileups might also be interpreted as an indicator of stored internal energy that can be recovered when a metal is heat treated. Therefore, as primary variables, we choose the total strain, $\boldsymbol{e}$, the plastic strain, $\boldsymbol{e}^{\mathrm{p}}$, dislocation density, $\tilde{\rho}^{\mathrm{d}}$, and the entropy, η. We assume the internal energy, U, per unit volume is a function of these variables

$$U = U(\eta, \boldsymbol{e}, \boldsymbol{e}^{\mathrm{p}}, \tilde{\rho}^{\mathrm{d}}). \tag{2.1}$$

With the assumptions of no heat source and no heat flux, the first law of thermodynamics reduces to

$$\dot{U} = \boldsymbol{\sigma} : \dot{\boldsymbol{e}}, \tag{2.2}$$

in which $\boldsymbol{\sigma}$ denotes the stress, and a superposed dot denotes a derivative with respect to time. The use of the chain rule results in

$$T\dot{\eta} = \left(\boldsymbol{\sigma} - \frac{\partial U}{\partial \boldsymbol{e}}\right) : \dot{\boldsymbol{e}} + D^{\mathrm{P}} \quad T \equiv \frac{\partial U}{\partial \eta}, \tag{2.3}$$

in which T is the temperature and D^{P}, the dissipation power, is defined by

$$D^{\mathrm{P}} = -\frac{\partial U}{\partial \boldsymbol{e}^{\mathrm{p}}} : \dot{\boldsymbol{e}}^{\mathrm{p}} - \frac{\partial U}{\partial \tilde{\rho}^{\mathrm{d}}} \dot{\tilde{\rho}}^{\mathrm{d}}. \tag{2.4}$$

The Clausius–Duhem form of the second law of thermodynamics is the inequality $T\dot{\eta} \geq 0$ for all strain rate. With the use of the first law and (2.3), the second law implies that

$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{e}} \quad D^{\mathrm{P}} \geq 0, \tag{2.5}$$

and the first law reduces to

$$T\dot{\eta} = D^{\mathrm{P}}. \tag{2.6}$$

As additional constitutive relations, we define the stress, $\boldsymbol{\sigma}^{\mathrm{p}}$, conjugate to plastic strain and the stress, σ^{d}, conjugate to dimensionless dislocation density to be

$$\boldsymbol{\sigma}^{\mathrm{p}} = -\frac{\partial U}{\partial \boldsymbol{e}^{\mathrm{p}}} \quad \sigma^{\mathrm{d}} = \frac{\partial U}{\partial \tilde{\rho}^{\mathrm{d}}}. \tag{2.7}$$

The difference in sign for the two stress variables provides conventional definitions later in the development. The dissipation power becomes

$$D^{\mathrm{P}} = P^{\mathrm{P}} - \sigma^{\mathrm{d}}\dot{\tilde{\rho}}^{\mathrm{d}} \quad P^{\mathrm{P}} \equiv \boldsymbol{\sigma}^{\mathrm{p}} : \dot{\boldsymbol{e}}^{\mathrm{p}}, \tag{2.8}$$

in which P^{P} denotes the plastic power. We emphasize the point that the dissipation power is not equal to the plastic power.

A conventional approach is to assume that the dependence of U on strain occurs only as the difference of the total and plastic strains (elastic strain), or

$$U(\eta, \boldsymbol{e}, \boldsymbol{e}^{\mathrm{p}}, \tilde{\rho}^{\mathrm{d}}) = U(\eta, \boldsymbol{e}^{\mathrm{e}}, \tilde{\rho}^{\mathrm{d}}) \quad \boldsymbol{e}^{\mathrm{e}} = \boldsymbol{e} - \boldsymbol{e}^{\mathrm{p}}. \tag{2.9}$$

The implication of this restriction and (2.7) is that $\boldsymbol{\sigma}^{\mathrm{p}} = \boldsymbol{\sigma}$

Next, to provide the framework for an uncoupled formulation, we postulate that the internal energy is separable into parts that involve the entropy, the elastic strain, and the dislocation density

$$U(\eta, \boldsymbol{e}^{\mathrm{e}}, \tilde{\rho}^{\mathrm{d}}) = U_{\eta}(\eta) + U_{e}(\boldsymbol{e}^{\mathrm{e}}) + U_{\rho}(\tilde{\rho}^{\mathrm{d}}), \tag{2.10}$$

in which the subscripts on the right side are used to indicate the independent variable for the respective contributions.

(*b*) *Entropy contribution*

For the entropy contribution to the internal energy, we assume the classical form of

$$U_{\eta}(\eta) = c_{\mathrm{v}} T_0 [e^{(\eta-\eta_0)/c_{\mathrm{v}}} - 1]. \tag{2.11}$$

The material constant, c_{v}, is the heat capacity at constant volume, and η_0 and T_0 are reference values for entropy and temperature, respectively. It follows that

$$T = T_0 e^{(\eta-\eta_0)/c_{\mathrm{v}}}, \tag{2.12}$$

and

$$T\dot{\eta} = c_{\mathrm{v}} \dot{T}. \tag{2.13}$$

If we work with T rather than η, we obtain the entropy and the thermal part of the internal energy as

$$\eta = \eta_0 + c_{\mathrm{v}} \ln \frac{T}{T_0} \quad U_{\eta}^{*}(T) = c_{\mathrm{v}}(T - T_0). \tag{2.14}$$

$U_{\eta}^{*}$is not a potential function since $\partial U_{\eta}^{*}/\partial T \neq \eta$ and, therefore, many choose not to use such a function.

Alternatively, we can perform a Taylor expansion on the exponential and retain only the lowest terms to obtain

$$\left.\begin{aligned} U_{\eta}(\eta) &= T_0\left[(\eta-\eta_0) + \frac{1}{2c_{\mathrm{v}}}(\eta-\eta_0)^2 + \cdots\right] \\ T &= T_0\left[1 + \frac{1}{c_{\mathrm{v}}}(\eta-\eta_0) + \cdots\right] \end{aligned}\right\}. \tag{2.15}$$

As long as the linear relationship is satisfactorily accurate, we can think of η as merely a scaled value of temperature.

(c) *Evolution of plasticity*

To describe the development of plastic deformation, we construct the mathematical model through the use of evolution equations

$$\dot{\boldsymbol{e}}^{\mathrm{p}} = \dot{\lambda}\boldsymbol{m}(\eta, \boldsymbol{e}^{\mathrm{e}}, \tilde{\rho}^{\mathrm{d}}) \quad \dot{\tilde{\rho}}^{\mathrm{d}} = \dot{\lambda} m^{\mathrm{d}}(\eta, \boldsymbol{e}^{\mathrm{e}}, \tilde{\rho}^{\mathrm{d}}), \tag{2.16}$$

in which λ is a monotonically increasing plasticity parameter and the evolution functions, $\boldsymbol{m}$ and m^{d}, are assumed to depend on the same parameters as U. Since η is uniquely related to T, the dependence on η can be replaced with T. The evolution function m^{d} serves to connect the development of plastic strain to the increase in dislocation density.

If an 'effective' stress, Σ^{eff}, and an 'effective' dislocation density, $\bar{\Sigma}^{\mathrm{d}}$, are defined by

$$\Sigma^{\mathrm{eff}} = \boldsymbol{\sigma}^{\mathrm{p}} : \boldsymbol{m} \quad \bar{\Sigma}^{\mathrm{d}} = \sigma^{\mathrm{d}} m^{\mathrm{d}}, \tag{2.17}$$

it follows from (2.8) that the plastic power and dissipation power are

$$P^{\mathrm{P}} = \dot{\lambda}\Sigma^{\mathrm{eff}} \quad D^{\mathrm{P}} = \dot{\lambda}\Sigma^{\mathrm{D}} \quad \Sigma^{\mathrm{D}} \equiv [\Sigma^{\mathrm{eff}} - \bar{\Sigma}^{\mathrm{d}}], \tag{2.18}$$

in which Σ^{D} is defined to be the 'effective' stress of dissipation. A thermodynamically consistent formulation must ensure that the dissipation power is positive semi-definite as indicated by (2.5). To this end one can define a yield function F based on the expression for the dissipation power to be

$$F = \Sigma^{\mathrm{D}} - F_{\mathrm{a}}, \tag{2.19}$$

in which F_{a} is an additive function (which could be a constant). The usual restrictions are made that plasticity does not occur if $F<0$, the state $F>0$ is not allowed, and plastic deformation may occur only when $F=0$ in which case $\Sigma^{\mathrm{D}} = F_{\mathrm{a}}$ and $D^{\mathrm{P}} = \dot{\lambda}F_{\mathrm{a}}$. The required inequality is satisfied if the restriction is imposed that $F_{\mathrm{a}} \geq 0$.

(d) *Implication for predictions of temperature*

For constant c_{v}, (2.6) and (2.13) indicate that the rate of increase of temperature is proportional to the dissipation power. It has become customary, primarily with regard to computational methods (e.g. Camacho & Ortiz 1997), to express the rate of increase of temperature as proportional to a factor, β, times the plastic power so that, by inference

$$\beta \equiv \frac{c_{\mathrm{v}}\dot{T}}{P^{\mathrm{p}}} = \frac{D^{\mathrm{P}}}{P^{\mathrm{P}}} = \frac{\Sigma^{\mathrm{D}}}{\Sigma^{\mathrm{eff}}} = 1 - \frac{\bar{\Sigma}^{\mathrm{d}}}{\Sigma^{\mathrm{eff}}}. \tag{2.20}$$

Therefore, β involves a ratio of effective stresses, and only under additional, and very restrictive, assumptions will β be a constant.

3. A specific model

(a) *Preliminary comments*

At this point, we begin to make additional simplifying assumptions in order to make specific conclusions concerning the form of the function β and the resulting implications concerning predictions for temperature. The objective is to be as straightforward as possible. Then, alternative formulations can be easily developed if some of the assumptions are changed.

(*b*) *Elastic contribution*

If isotropic linear elasticity is assumed, then the expression for the elastic strain energy is

$$U_e(\boldsymbol{e}^{\mathrm{e}}) = \tfrac{1}{2}[3B\boldsymbol{e}^{\mathrm{e}} : \boldsymbol{P}^{\mathrm{s}} : \boldsymbol{e}^{\mathrm{e}} + 2G\boldsymbol{e}^{\mathrm{e}} : \boldsymbol{P}^{\mathrm{d}} : \boldsymbol{e}^{\mathrm{e}}], \tag{3.1}$$

where B and G denote the bulk and shear modulus, respectively. The spherical projection, $\boldsymbol{P}^{\mathrm{s}}$, and deviatoric projection, $\boldsymbol{P}^{\mathrm{d}}$, are defined by

$$\boldsymbol{P}^{\mathrm{s}} = \tfrac{1}{3}\boldsymbol{i}\otimes\boldsymbol{i} \quad \boldsymbol{P}^{\mathrm{d}} = \boldsymbol{I} - \boldsymbol{P}^{\mathrm{s}}, \tag{3.2}$$

with $\boldsymbol{i}$ the second-order identity, and $\boldsymbol{I}$ the symmetric fourth-order identity tensors. The constitutive equations for stress and plastic conjugate stress become

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{p}} = 3B\boldsymbol{P}^{\mathrm{s}} : \boldsymbol{e}^{\mathrm{e}} + 2G\boldsymbol{P}^{\mathrm{d}} : \boldsymbol{e}^{\mathrm{e}}. \tag{3.3}$$

For the moment, we leave open the form of the dislocation energy function.

(*c*) *von Mises plasticity*

If $\boldsymbol{\sigma}^{\mathrm{dv}}$ denotes the stress deviator, then von Mises plasticity is obtained by choosing the first evolution function of (2.16) to be

$$\boldsymbol{m} = \sqrt{\frac{3}{2}}\frac{\boldsymbol{\sigma}^{\mathrm{dv}}}{(\boldsymbol{\sigma}^{\mathrm{dv}} : \boldsymbol{\sigma}^{\mathrm{dv}})^{1/2}}. \tag{3.4}$$

The result is that

$$\Sigma^{\mathrm{eff}} = \left(\tfrac{3}{2}\boldsymbol{\sigma}^{\mathrm{dv}} : \boldsymbol{\sigma}^{\mathrm{dv}}\right)^{1/2} \quad \dot{\lambda} = \dot{\bar{e}}^{\mathrm{p}} \equiv \frac{(\dot{\boldsymbol{e}}^{\mathrm{p}} : \dot{\boldsymbol{e}}^{\mathrm{p}})^{1/2}}{(\boldsymbol{m} : \boldsymbol{m})^{1/2}} = \left(\tfrac{2}{3}\dot{\boldsymbol{e}}^{\mathrm{p}} : \dot{\boldsymbol{e}}^{\mathrm{p}}\right)^{1/2}. \tag{3.5}$$

Note that the effective plastic strain, $\bar{e}^{\mathrm{p}}$, and λ can be used interchangeably.

A von Mises yield function, F^{vM}, is typically chosen to be of the form

$$F^{\mathrm{vM}} = \Sigma^{\mathrm{eff}} - \sigma^{\mathrm{y}}, \tag{3.6}$$

with the function σ^{y} denoting the flow stress. It is observed that the flow stress of most metals varies as the square root of dislocation density (Follansbee & Kocks (1988)), or

$$\sigma^{\mathrm{y}} = \sigma_0^{\mathrm{y}} + \sigma_{\mathrm{h}}^{\mathrm{y}} \qquad \sigma_{\mathrm{h}}^{\mathrm{y}} = \sigma_{\mathrm{m}}^{\mathrm{y}}\sqrt{\tilde{\rho}^{\mathrm{d}}}. \tag{3.7}$$

We interpret σ_0^{y} to be the yield stress associated with a perfectly annealed condition ($\tilde{\rho}^{\mathrm{d}} = 0$) and $\sigma_{\mathrm{m}}^{\mathrm{y}}$ to be the maximum possible enhancement to the annealed yield stress that occurs when $\tilde{\rho}^{\mathrm{d}} = 1$.

(*d*) *Strain hardening*

For metals, the hardening part of the yield stress is also often given as a function of effective plastic strain as follows

$$\sigma^{\mathrm{y}} = \sigma_0^{\mathrm{y}} + \sigma_{\mathrm{h}}^{\mathrm{y}} \qquad \sigma_{\mathrm{h}}^{\mathrm{y}} = \sigma_{\mathrm{m}}^{\mathrm{y}}\tanh\zeta\bar{e}^{\mathrm{p}}, \tag{3.8}$$

with the material parameter ζ chosen to provide a fit to experimental data. Of course, alternative hardening functions can be chosen. The initial yield and maximum enhancement stresses have the same connotations as used in (3.7). The two forms for $\boldsymbol{\sigma}^{\mathrm{y}}$ given in (3.7) and (3.8) yield

$$\tanh \zeta \bar{e}^{\mathrm{p}} = \sqrt{\tilde{\rho}^{\mathrm{d}}}. \tag{3.9}$$

By taking a derivative, utilising the equivalence of λ and $\bar{e}^{\mathrm{p}}$, and using the definition of m^{d} based on (2.16), we obtain

$$m^{\mathrm{d}} = 2\zeta\sqrt{\tilde{\rho}^{\mathrm{d}}}(1-\tilde{\rho}^{\mathrm{d}}). \tag{3.10}$$

If desired, this function can be transformed to one that depends on effective plastic strain.

(*e*) *Internal energy associated with dislocations*

Here, a specific form for the contribution to the internal energy designated as U_ρ in (2.10) is proposed. Rosakis *et al.* (2000) provide a plot of experimental data relating this term (called stored energy of cold work) to effective plastic strain for aluminum 2024-T351. The data suggest that for the given range of effective plastic strain, a linear assumption would be entirely reasonable. However, it also seems plausible to assume that with large strain this energy should saturate to a limiting value. Since the hyperbolic tangent is linear for small values of its argument, we use (3.9) and suggest that the following simple form might be appropriate for this material and, perhaps, for other materials

$$U_\rho = C_{\mathrm{d}}\sqrt{\tilde{\rho}^{\mathrm{d}}}, \tag{3.11}$$

in which C_{d} is a material constant. It follows from (2.7) that the conjugate dislocation stress is

$$\sigma^{\mathrm{d}} = \frac{C_{\mathrm{d}}}{2\sqrt{\tilde{\rho}^{\mathrm{d}}}}, \tag{3.12}$$

and (2.17) provides the ‘effective’ dislocation density

$$\bar{\Sigma}^{\mathrm{d}} = \zeta C_{\mathrm{d}}(1-\tilde{\rho}^{\mathrm{d}}). \tag{3.13}$$

The use of (2.18), (2.19), (3.6) and (3.7) yields the following expression for the additive part of the yield function

$$F_{\mathrm{a}} = \sigma^{\mathrm{y}} - \bar{\Sigma}^{\mathrm{d}} = \sigma_0^{\mathrm{y}} + \sigma_{\mathrm{m}}^{\mathrm{y}}\sqrt{\tilde{\rho}^{\mathrm{d}}} - \zeta C_{\mathrm{d}}(1-\tilde{\rho}^{\mathrm{d}}). \tag{3.14}$$

The dissipation requirement that F_{a} be positive imposes the inequality $\sigma_0^{\mathrm{y}} \geq \zeta C_{\mathrm{d}}$. In general, whenever values for the material constants are selected, the inequality on F_{a} should be checked for all $\tilde{\rho}^{\mathrm{d}}$.

(*f*) *Implications for temperature predictions*

The use of $F^{\mathrm{vM}}=0$ and (3.6) implies $\Sigma^{\mathrm{eff}} = \sigma^{\mathrm{y}}$ so that (2.20) reduces to

$$\beta = \frac{\sigma^{\mathrm{y}} - \bar{\Sigma}^{\mathrm{d}}}{\sigma^{\mathrm{y}}} = \frac{F_{\mathrm{a}}}{\sigma^{\mathrm{y}}}, \tag{3.15}$$

in which (3.14) has been used. The inequality, $F_{\mathrm{a}} \geq 0$, ensures that $0 \leq \beta \leq 1$, as should be expected. If the yield stress and β are available as functions of effective plastic strain from experimental data, the effective stress $\bar{\Sigma}^{\mathrm{d}}$ as a function of effective plastic strain (or dislocation density) can be determined. One can then work back to derive the corresponding forms for the conjugate stress and internal energy. On the other hand, if we accept the assumed form for the internal energy given in (3.11), then

$$\beta = 1 - \frac{\zeta C_{\mathrm{d}}(1 - \tilde{\rho}^{\mathrm{d}})}{\sigma_0^{\mathrm{y}} + \sigma_{\mathrm{m}}^{\mathrm{y}} \sqrt{\tilde{\rho}^{\mathrm{d}}}}. \tag{3.16}$$

If the dislocation state is saturated ($\tilde{\rho}^{\mathrm{d}} = 1$), then $\beta = 1$. The other limiting case of the perfectly annealed state ($\tilde{\rho}^{\mathrm{d}} = 0$) yields the value

$$\beta_0 \equiv \beta|_{\tilde{\rho}^{\mathrm{d}}=0} = 1 - \frac{\zeta C_{\mathrm{d}}}{\sigma_0^{\mathrm{y}}}. \tag{3.17}$$

The inequality $\sigma_0^{\mathrm{y}} \geq \zeta C_\rho$ implies that $0 \leq \beta_0 \leq 1$. If experimental data for U_ρ are not available, but β_0 is experimentally available, then (3.17) can be used to obtain C_{d}. In order to provide plots of representative results in the next section a value for β_0 has been assumed.

(g) Associativity

With the use of (3.5), (3.12) and $\boldsymbol{\sigma}^{\mathrm{p}} = \boldsymbol{\sigma}$, the yield function of (3.6) can be expressed in terms of the conjugate stresses

$$F^{\mathrm{vM}}(\boldsymbol{\sigma}^{\mathrm{p}}, \sigma^{\mathrm{d}}) = \Sigma^{\mathrm{eff}} - \sigma_0^{\mathrm{y}} - \sigma_{\mathrm{m}}^{\mathrm{y}} \frac{C_{\mathrm{d}}}{2\sigma^{\mathrm{d}}}. \tag{3.18}$$

The evolution equations of (2.16) are partially associated in the sense that

$$\boldsymbol{m} = \frac{\partial F^{\mathrm{vM}}}{\partial \boldsymbol{\sigma}^{\mathrm{p}}} \quad m^{\mathrm{d}} \neq \frac{\partial F^{\mathrm{vM}}}{\partial \sigma^{\mathrm{d}}}. \tag{3.19}$$

However, the symmetry of the tangent tensor depends only on the satisfaction of the first relationship so the fact that the inequality holds for m^{d} is not considered to be detrimental to the theory.

(h) Dislocation internal energy, plastic work and dissipation

Because of the uncoupled forms for the energy and the yield function, we can perform integrations to obtain explicit expressions for the dissipation and the plastic work. The plastic work, W^{P}, and dissipation, D, are defined as the time integrals of the plastic power and dissipation power, respectively

$$W^{\mathrm{P}} = \int P^{\mathrm{P}} \, \mathrm{d}t \quad D = \int D^{\mathrm{P}} \, \mathrm{d}t. \tag{3.20}$$

Recall that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{p}}$ because of the restricted form for the internal energy. With the use of (2.8), (2.17), (2.18), (3.5) and the von Mises yield condition (3.6), various integral expressions for the plastic work are obtained as follows

$$W^{\mathrm{P}} = \int \Sigma^{\mathrm{eff}} \, \mathrm{d}\lambda = \int \sigma^{\mathrm{y}} \, \mathrm{d}\lambda = \int [\sigma_0^{\mathrm{y}} + \sigma_{\mathrm{m}}^{\mathrm{y}} \tanh \zeta \bar{e}^{\mathrm{p}}] \mathrm{d}\bar{e}^{\mathrm{p}}. \tag{3.21}$$

The latter one yields

$$W^{\mathrm{P}} = \sigma_0^{\mathrm{y}} \bar{e}^{\mathrm{p}} + \frac{\sigma_{\mathrm{m}}^{\mathrm{y}}}{\zeta} \ln(\cosh \zeta \bar{e}^{\mathrm{p}}). \tag{3.22}$$

With regard to the dissipation, the first of (2.8) becomes simply

$$D = W^{\mathrm{P}} - U_{\rho}, \tag{3.23}$$

which states that the dissipation is simply the difference between the plastic work and the dislocation internal energy.

The uncoupled formulation leads to specific forms for plastic work and total dissipation so it is appropriate here to speak of dissipation rate and rate of plastic work rather than dissipation power and plastic power, respectively. This coalescence of terminology does not hold in general.

(*i*) *An evolution difficulty*

Although the theory developed so far is complete, there is one difficulty that arises when the model is implemented numerically. The problem is that the evolution function m^{d} given in (3.10) is zero when the dislocation density is zero, and if deformation begins at an annealed state, there is no way for the dislocation density to evolve unless m^{d} is artificially assigned a small, positive initial value. Here we show that a simple change of variable can rectify the situation. Let

$$\rho * = \sqrt{\tilde{\rho}^{\mathrm{d}}}. \tag{3.24}$$

Then

$$\sigma_{\mathrm{h}}^{\mathrm{y}} = \sigma_{\mathrm{m}}^{\mathrm{y}} \, \rho * \quad \tanh \zeta \bar{e}^{\mathrm{p}} = \rho * . \tag{3.25}$$

Consequently, we obtain

$$\dot{\rho} * = m^{\mathrm{d}*} \dot{\lambda}, \tag{3.26}$$

where

$$m^{\mathrm{d}*} = \zeta(1 - \rho^*). \tag{3.27}$$

Now the evolution function $m^{\mathrm{d}*}$ is not zero if the initial value of ρ^* is zero, and the evolution of ρ^* ceases as ρ^* approaches the saturation value of unity. We continue to utilise the dislocation density in our formulation but for numerical calculations, the transformation to the new variable may be more useful.

(*j*) *Rate effects*

There are numerous methods for combining strain-rate effects with plasticity. These include: (i) the overstress model of Perzyna (1966) in which the plastic strain rate depends on the amount an effective stress exceeds the yield stress, (ii) the addition of a strain obtained from a rate-dependent constitutive equation similar in form to that of plasticity but with $\dot{\lambda}$ replaced with a material constant, (iii) the use of a yield function that depends on either total strain rate (Rubin 1982) or plastic strain rate (Di Melfi & Kramer 1980), (iv) and the addition of a viscous stress. Since we are interested only in providing a basic framework in a thermodynamical setting, we choose one that appears to be widely used, namely; modify the yield function to include plastic strain rate. Specifically, we do this by changing the hardening part of the flow stress through a positive function, g, to reflect the enhancement of yield with plastic strain rate as follows

$$\sigma^{\mathrm{y}}(\bar{e}^{\mathrm{p}}, \dot{\bar{e}}^{\mathrm{p}}) = \sigma_0^{\mathrm{y}} + \sigma_{\mathrm{h}}^{\mathrm{y}}(\bar{e}^{\mathrm{p}}, \dot{\bar{e}}^{\mathrm{p}}) \quad \sigma_{\mathrm{h}}^{\mathrm{y}} = \sigma_{\mathrm{m}}^{\mathrm{y}}[1 + g(\dot{\bar{e}}^{\mathrm{p}})]\tanh \zeta \bar{e}^{\mathrm{p}} \quad g(0) = 0. \tag{3.28}$$

The initial-value restriction is placed on g so that the formulation reverts to the existing one when the effective plastic strain rate is zero.

The rate-independent formulation for dissipation power and plastic power continue to hold with this modified form of the flow stress. If, for example, a path of constant strain rate is followed, one merely replaces $\sigma_{\mathrm{m}}^{\mathrm{y}}$ with the new constant $\sigma_{\mathrm{m}}^{\mathrm{y}}[1 + g(\dot{\bar{e}}^{\mathrm{p}})]$ in the expression for β given in (3.16). The result is that strain rate can only increase β with the inequality $0 \le \beta \le 1$ always satisfied. If warranted by experimental data or microstructural arguments, the parameters σ_0^{y} and C_{d} could be allowed to depend on strain rate as well.

We will not pursue strain-rate effects any further except to say that most paths have variable strain rate. Also, for modern flow-stress models that reflect path dependence with respect to strain-rate (Chen & Gray 1966; Follansbee & Kocks 1988), explicit forms for plastic work and total dissipation do not exist so that it is then incorrect to refer to plastic power and dissipation power as rates of plastic work and dissipation, respectively.

(*k*) *Concluding comments*

In this section we have provided a small, but important, extension to the elementary model of von Mises plasticity. The new contribution is a proposed form for an uncoupled addition to the internal energy that depends on dislocation density. The result provides a logical way to describe the β-effect where β is derived as a function of deformation rather than being assumed constant. The result yields particularly simple formulas for determining temperature under adiabatic conditions. If the assumed form for internal energy proves to be inadequate for application to a specific material, the elementary framework that is provided allows a user to easily make modifications.

For example, there are experimental data that indicate β may initially decrease from a value close to unity at zero dislocation density, and then increase with a monotonic increase in effective plastic strain. In fact for an annealed copper, it is plausible to argue that the beginning value for β should be one. It is rather straightforward to show that such a feature is exhibited by selecting an internal energy contribution of the form $(\tilde{\rho}^{\mathrm{d}})^{3/2}$ rather than the square root of the dislocation density used in the illustrative development.

In §4, we recapitulate the equations in dimensionless form with this extra term included. Sample plots are then provided to illustrate the general features of the formulation.

4. Temperature predictions based on the uncoupled theory

(*a*) *Dimensionless variables*

It is good practice from both theoretical and numerical viewpoints to introduce dimensionless variables which will be denoted with an overscribed *tilda*, a notation already used in connection with dislocation density. We use the process of defining dimensionless variables also as an opportunity to summarize the set of governing equations. Since the theory is rate independent, there is no need to introduce a dimensionless time.First, we define dimensionless entropy and temperature

$$\tilde{\eta} = \frac{\eta - \eta_0}{c_\mathrm{v}} \quad \tilde{T} = \frac{T - T_0}{T_0}. \tag{4.1}$$

We choose the reference temperature to be $T_0 = 300$ K and the reference entropy, η_0, is unspecified. Next, we divide all stress-related variables by the initial yield stress to obtain corresponding dimensionless terms

$$\left.\begin{aligned}
\tilde{\boldsymbol{\sigma}} &= \frac{\boldsymbol{\sigma}}{\sigma_0^\mathrm{y}} & \tilde{\sigma}^\mathrm{d} &= \frac{\sigma^\mathrm{d}}{\sigma_0^\mathrm{y}} \\
\tilde{\Sigma}^\mathrm{eff} &= \frac{\Sigma^\mathrm{eff}}{\sigma_0^\mathrm{y}} & \tilde{\bar{\Sigma}}^\mathrm{d} &= \frac{\bar{\Sigma}^\mathrm{d}}{\sigma_0^\mathrm{y}} \\
\tilde{\Sigma}^\mathrm{D} &= \frac{\Sigma^\mathrm{D}}{\sigma_0^\mathrm{y}} & \tilde{C}_\mathrm{d} &= \frac{C_\mathrm{d}}{\sigma_0^\mathrm{y}} \\
\tilde{B} &= \frac{B}{\sigma_0^\mathrm{y}} & \tilde{G} &= \frac{G}{\sigma_0^\mathrm{y}} \\
\tilde{\sigma}_\mathrm{m}^\mathrm{y} &= \frac{\sigma_\mathrm{m}^\mathrm{y}}{\sigma_0^\mathrm{y}} & \tilde{\sigma}^\mathrm{y} &= \frac{\sigma^\mathrm{y}}{\sigma_0^\mathrm{y}}
\end{aligned}\right\}. \tag{4.2}$$

The dimensionless entropy and elastic internal energies are

$$\left.\begin{aligned}
\tilde{U}_\eta &= \frac{U_\eta(\eta)}{\sigma_0^\mathrm{y}} = \frac{1}{r_\mathrm{c}^\mathrm{y}}[e\tilde{\eta} - 1] \\
\tilde{U}_\mathrm{e} &= \frac{U_\mathrm{e}(\boldsymbol{e}^\mathrm{e})}{\sigma_0^\mathrm{y}} = \tfrac{1}{2}[3\tilde{B}\boldsymbol{e}^\mathrm{e} : \boldsymbol{P}^\mathrm{s} : \boldsymbol{e}^\mathrm{e} + 2\tilde{G}\boldsymbol{e}^\mathrm{e} : \boldsymbol{P}^\mathrm{d} : \boldsymbol{e}^\mathrm{e}],
\end{aligned}\right. \tag{4.3}$$

in which we have introduced the ratio of the initial yield stress to a thermal reference stress

$$r_\mathrm{c}^\mathrm{y} = \frac{\sigma_0^\mathrm{y}}{c_\mathrm{v} T_0}. \tag{4.4}$$

Two contributions are now assumed for the dislocation internal energy; the first term is the form suggested in (3.11) and, as shown later, the second term provides more flexibility in an attempt to have the function β reflect features exhibited by experimental data

$$\tilde{U}_\rho = \frac{U_\rho(\tilde{\rho}^\mathrm{d})}{\sigma_0^\mathrm{y}} = \tilde{C}_\mathrm{d1}(\tilde{\rho}^\mathrm{d})^{1/2} + \tilde{C}_\mathrm{d2}(\tilde{\rho}^\mathrm{d})^{3/2}. \tag{4.5}$$

Dimensionless forms of plastic power and dissipation power given in (2.18) become

$$\tilde{P}^\mathrm{P} \equiv \frac{P^\mathrm{P}}{\sigma_0^\mathrm{y}} = \dot{\lambda}\tilde{\Sigma}^\mathrm{eff} \quad \tilde{D}^\mathrm{P} \equiv \frac{D^\mathrm{P}}{\sigma_0^\mathrm{y}} = \dot{\lambda}\tilde{\Sigma}^\mathrm{D} \quad \tilde{\Sigma}^\mathrm{D} = \tilde{\Sigma}^\mathrm{eff} - \tilde{\bar{\Sigma}}^\mathrm{d}. \tag{4.6}$$

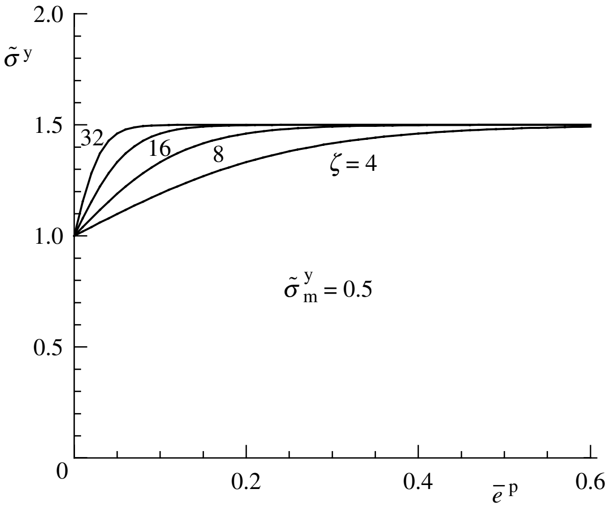

Figure 1. Dimensionless yield stress as a function of effective plastic strain.

(*b*) *Representative plots*

The hardening rule for flow stress from (3.8) and (3.9) becomes

$$\tilde\sigma^{y} = 1 + \tilde\sigma^{y}_{h} \quad \tilde\sigma^{y}_{h} = \tilde\sigma^{y}_{m}\sqrt{\tilde\rho^{d}} = \tilde\sigma^{y}_{m}\tanh\zeta\bar e^{p}. \tag{4.7}$$

The effect of various choices of ζ for describing the hardening function is illustrated in figure 1. Large values for ζ correspond to rapid hardening with an increase in effective plastic strain.

With the use of (2.7), (2.18), (3.9) and (3.10), the effective dislocation density becomes

$$\tilde{\bar\Sigma}^{d} = \zeta[1-\tanh^{2}\zeta\bar e^{p}][\tilde C_{d1} + 3\tilde C_{d2}\tanh^{2}\zeta\bar e^{p}], \tag{4.8}$$

while (3.14) and (3.15) yield the following expression

$$\beta = \frac{\tilde F_{a}}{\tilde\sigma^{y}} \quad \tilde F_{a} \equiv \tilde\sigma^{y} - \tilde{\bar\Sigma}^{d}, \tag{4.9}$$

where $\tilde F_{a}$ is the additional part of the yield function, in dimensionless form. If β_0 and ζ are considered prescribed values, then the parameter $\tilde C_{d1}$ must be chosen to satisfy

$$\tilde C_{d1} = (1-\beta_0)/\zeta. \tag{4.10}$$

Plots of β showing the effects of dimensionless material parameters are given in figures 2–4. First, we set $\tilde C_{d2}=0$ so only the effect of a dislocation internal energy of the form $\tilde U_{\rho} = \tilde C_{d1}(\tilde\rho^{d})^{1/2}$ is considered. The effect of ζ on β for $\tilde\sigma^{y}_{m}$ is shown in figure 2*a* while the effect of $\tilde\sigma^{y}_{m}$ on β for $\zeta=8$ is given in figure 2*b*. As expected, β increases monotonically with effective plastic strain from the initial value β_0 to the maximum value of 1.

Next, the effect of dislocation internal energy of the form $\tilde U_{\rho} = \tilde C_{d2}(\tilde\rho^{d})^{3/2}$ is illustrated. It is possible to find the equivalent plastic strain at which the minimum value for β occurs and then choose $\tilde C_{d2}$ to satisfy a prescribed minimum value obtained from experimental data. Instead, the following

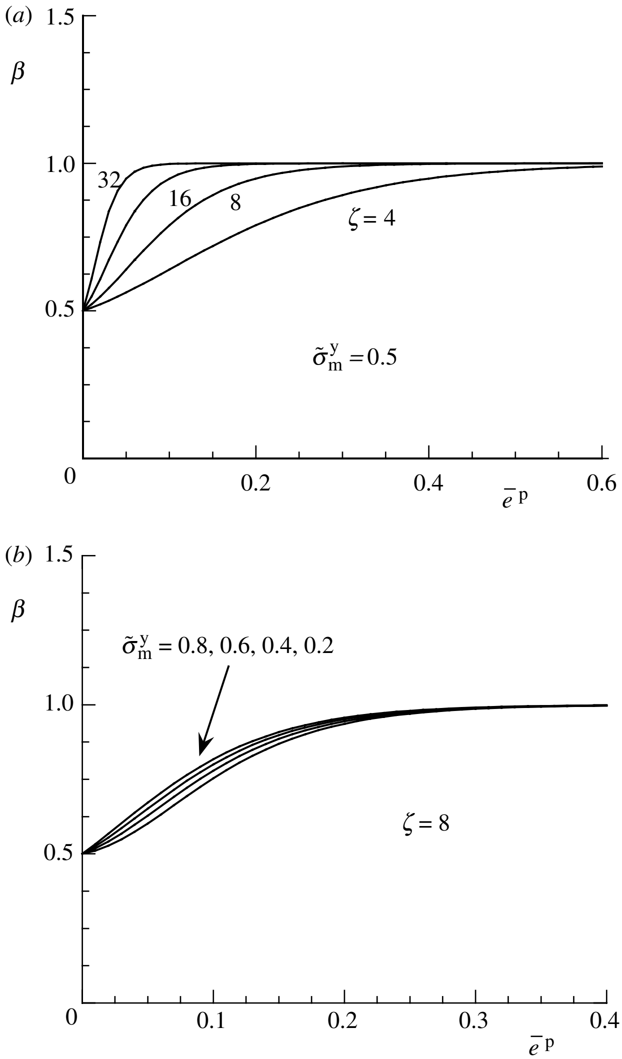

Figure 2. Plot of β versus $\bar{e}^{\rm p}$ for $\beta_0 = 0.5$ and $\tilde{C}_{\rm d2} = 0$. (*a*) The effect of ζ with $\tilde{\sigma}^{\rm y}_{\rm m}$ fixed. (*b*) The effect of $\tilde{\sigma}^{\rm y}_{\rm m}$ with ζ fixed.

expression for $\tilde{C}_{\rm d2}$, selected in an *ad hoc* manner, is used

$$\tilde{C}_{\rm d2} = \frac{10\tilde{\sigma}^{\rm y}_{\rm m}}{\zeta(1 + 10\tilde{\sigma}^{\rm y}_{\rm m})}. \tag{4.11}$$

This choice, which provides approximately the same minimum value for β for a wide range of cases, is used for all examples where $\tilde{C}_{\rm d2}$ is not zero. We emphasize that the relation (4.11) is chosen merely to provide general features of the model; experimental data may require a substantially different form. Figure 3 provides plots of β with the initial value $\beta_0 = 1$ with the result from (4.10) that $\tilde{C}_{\rm d1} = 0$. Only variations in ζ are shown as very little change with $\tilde{\sigma}^{\rm y}_{\rm m}$ is observed. The basic feature is a reduction from the initial value to a minimum and then an increase back to unity as the effective plastic strain becomes large.

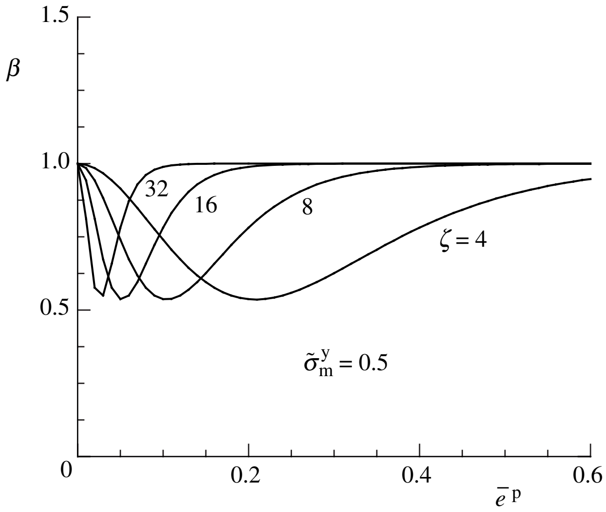

Figure 3. Plot of β versus $\bar{e}^{\mathrm{p}}$ for $\beta_0 = 1.0(\tilde{C}_{\mathrm{d1}} = 0)$ and $\tilde{C}_{\mathrm{d2}}$ given by (4.11).

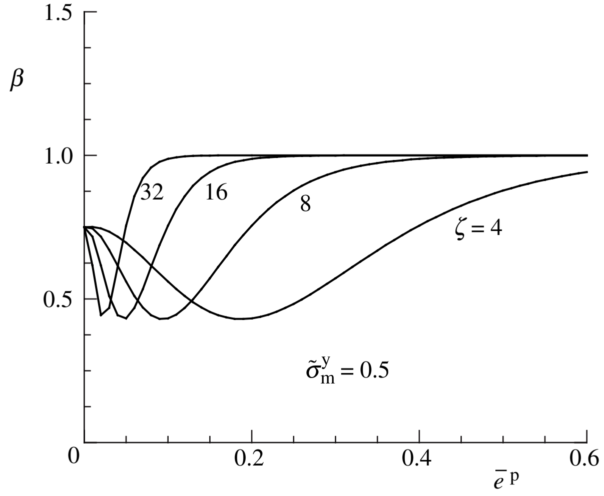

Figure 4. Plot of β versus $\bar{e}^{\mathrm{p}}$ for $\beta_0 = 0.75$ with $\tilde{C}_{\mathrm{d1}}$ and $\tilde{C}_{\mathrm{d2}}$ given by (4.10) and (4.11), respectively.

Finally, figure 4 shows representative distributions of β when both $\tilde{C}_{\mathrm{d1}}$ and $\tilde{C}_{\mathrm{d2}}$ are non-zero and given by (4.10) and 4.11), respectively, with $\beta_0 = 0.75$. Again, there is very little change with $\tilde{\sigma}^{\mathrm{y}}_{\mathrm{m}}$ so such a plot is not given. If experimental data reflect an even richer structure for β then the form for $\tilde{U}_\rho$ must be altered from that used here (Rosakis *et al.* 2000).

With β as a known function, we use (2.18), (2.20), (3.5), (4.4) and the yield condition $\boldsymbol{\Sigma}^{\mathrm{eff}} = \boldsymbol{\sigma}^{\mathrm{y}}$ to obtain

$$\dot{\tilde{T}} = \beta r^{\mathrm{y}}_{\mathrm{c}} \tilde{\sigma}^{\mathrm{y}} \dot{\bar{e}}^{\mathrm{p}}. \tag{4.12}$$

A simple integration algorithm is all that is needed to obtain temperature as a function of inelastic plastic strain. We see that a key factor in determining how

Table 1. *Table of material properties for certain metals at $T_0 = 300$ K*

material	mass density (ρ) (kg m^{-3})	specific heat (c_ρ) (J kg^{-1} K^{-1})	thermal capacity ($c_\mathrm{v} = \rho c_\rho$) (J m^{-3} K^{-1})	yield stress (σ_0^y) (M Pa)	young's modulus $\boldsymbol{E}_0$ (G Pa)	coefficient of thermal expansion α (1 k^{-1})	ratio of E to yield stress $\tilde{\boldsymbol{E}}_0 = \boldsymbol{E}_0/\boldsymbol{\sigma}_0^\mathrm{y}$	coefficient of thermal expansion $\tilde{\alpha} = \alpha T_0$	ratio of yield stress to thermal reference stress $\boldsymbol{r}_\mathrm{c}^\mathrm{y} = \boldsymbol{\sigma}_0^\mathrm{y}/\boldsymbol{c}_\mathrm{v}\boldsymbol{T}_0$
	(*a*) dimensional material parameters						(*b*) dimensionless material parameters		
copper	8890	393	3.52×10^6	150	110	17×10^{-6}	0.73×10^3	5.1×10^{-3}	0.142
aluminum	2700	887	2.39×10^6	130	69	24×10^{-6}	0.53×10^3	7.2×10^{-3}	0.181
steel	7800	460	3.59×10^6	250	200	12×10^{-6}	0.80×10^3	3.6×10^{-3}	0.232
tantalum	16 600	150	2.49×10^6	600	184	6.6×10^{-6}	0.31×10^3	2.0×10^{-2}	0.800

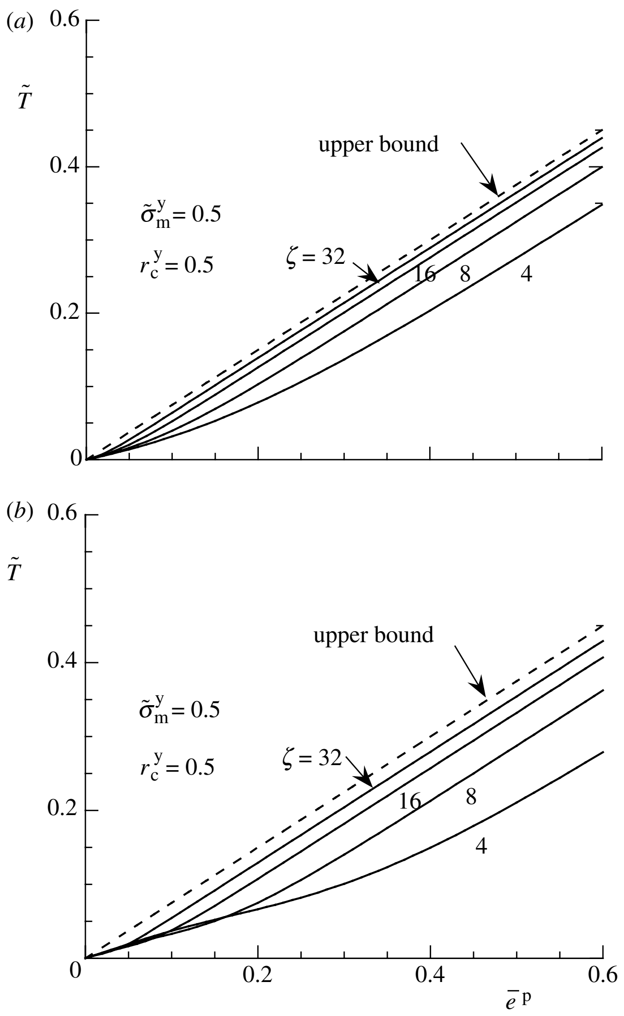

Figure 5. Dimensionless temperature versus effective plastic strain. (*a*) $\beta_0 = 0.5$ and $\tilde{C}_{d2} = 0$. (*b*) $\beta_0 = 0.75$ with $\tilde{C}_{d1}$ and $\tilde{C}_{d2}$ given by (4.10) and (4.11), respectively.

rapidly temperature increases with plastic deformation is the dimensionless material parameter r^{y}_{c}. Values of r^{y}_{c} are given in table 1 for some typical materials. It appears that the inequality $0.1 < r^{y}_{c} < 0.9$ includes the majority of cases.

An upper bound for temperature predictions is obtained simply by choosing $\beta = 1$ and replacing $\tanh \zeta \bar{e}^{p}$ with unity (essentially using the maximum value of the flow stress)

$$\tilde{T}^{ub} = r^{y}_{c}(1 + \tilde{\sigma}^{y}_{m})\bar{e}^{p}. \tag{4.13}$$

With $r^{y}_{c} = 0.5$ and $\tilde{\sigma}^{y}_{m} = 0.5$, plots of temperature obtained from (4.12) and the upper bound given by (4.13) are shown in figure 5*a* for $\beta_0 = 0.5$ with $\tilde{C}_{d2} = 0$, and in figure 5*b* for $\beta_0 = 0.75$ with $\tilde{C}_{d2}$ obtained from (4.11).

5. Coupled thermal and plasticity effects

(*a*) *Basic formulation*

We have considered so far an uncoupled thermo-mechanical theory that provides a very simple relationship for estimating increases in temperature. However, most material moduli depend on temperature so they are actually functions. If this dependence is taken into account, the result is a coupled theory that we consider next. In particular, we are interested in the implications of the presence of the dislocation internal energy.For the sake of simplicity and practical utility, we continue to eschew generality. Even so, we believe that the formulation inherently provides guidelines as to how a more general theory might be developed if warranted. Therefore, we do not consider anisotropy, large strains, large excursions in temperature and complex material dependence on temperature. However, even with these restrictions, the theory may reflect general trends that are physically meaningful.

We start the development of a consistent theory by modifying the uncoupled formulation and postulate a form for the internal energy that is slightly more general than the previous form

$$U(\eta, \boldsymbol{e}^{\mathrm{e}}, \rho^{\mathrm{d}}) = U_\eta(\eta) + U_{\mathrm{e}}(\eta, \boldsymbol{e}^{\mathrm{e}}) + U_\rho(\eta, \rho^{\mathrm{d}}). \tag{5.1}$$

We retain the form of (2.11) for the entropy internal energy by assuming the parameter c_{v} remains constant. We allow the elastic and dislocation internal energies to depend on entropy with specific forms to be given later. We decompose the temperature into two parts as follows

$$T = T_\eta + T_{\mathrm{a}} \quad T_\eta = \frac{\partial U_\eta}{\partial \eta} = T_0 e^{(\eta - \eta_0)/c_{\mathrm{v}}} \quad T_{\mathrm{a}} = \frac{\partial U_{\mathrm{e}}}{\partial \eta} + \frac{\partial U_\rho}{\partial \eta}. \tag{5.2}$$

We assume that the additional temperature is insignificant in comparison with the entropy temperature, i.e.

$$T_{\mathrm{a}} \ll T_\eta \quad \text{or} \quad T \approx T_\eta. \tag{5.3}$$

Now, the first of (2.14) can be used to express entropy in terms of temperature so that the elastic and dislocation density part of the internal energies can be expressed equally well as functions of temperature rather than entropy. The chain rule is used to obtain the following expression for the additional temperature to see later if the inequality of (5.3) is actually satisfied

$$T_{\mathrm{a}} = \frac{T}{c_{\mathrm{v}}} \frac{\partial}{\partial T} [U_{\mathrm{e}}(T, \boldsymbol{e}^{\mathrm{e}}) + U_\rho(T, \rho^{\mathrm{d}})]. \tag{5.4}$$

For convenience, we use the following dimensionless form for the additional temperature

$$\tilde{T}_{\mathrm{a}} = \frac{T_{\mathrm{a}}}{T_0}. \tag{5.5}$$

In the elastic part of the internal energy, we allow the moduli, G and B, to depend on temperature. In addition, we introduce a thermal strain, $\boldsymbol{e}^{\mathrm{T}}$, so that (3.1) is replaced with

$$\left.\begin{aligned} &U_{\mathrm{e}}(T, \boldsymbol{e}^{\mathrm{e}}) = \tfrac{1}{2}[3B(T)\boldsymbol{e}^{\mathrm{e}} : \boldsymbol{P}^{\mathrm{s}} : \boldsymbol{e}^{\mathrm{e}} + 2G(T)\boldsymbol{e}^{\mathrm{e}} : \boldsymbol{P}^{\mathrm{d}} : \boldsymbol{e}^{\mathrm{e}}] \\ &\boldsymbol{e}^{\mathrm{e}} = \boldsymbol{e} - \boldsymbol{e}^{\mathrm{p}} - \boldsymbol{e}^{\mathrm{T}}(T). \end{aligned}\right\} \tag{5.6}$$

As specific forms for thermal dependence using the dimensionless temperature of (4.1), we choose the following

$$B = B_0 e^{-\tilde{k}\tilde{T}} \quad G = G_0 e^{-\tilde{k}\tilde{T}} \quad \boldsymbol{e}^T = \alpha T_0 \tilde{T} \boldsymbol{i}, \tag{5.7}$$

in which $\tilde{k}$ is a dimensionless material constant representing a measure of how quickly the elastic moduli decay with temperature. B_0 and G_0 are the bulk and shear moduli, respectively, at the reference temperature, and α denotes the usual coefficient of thermal expansion and is assumed to be a constant. Conversions to dimensionless forms follow

$$\tilde{B}_0 = \frac{B_0}{\sigma_0^{\mathrm{y}}} \quad \tilde{G}_0 = \frac{G_0}{\sigma_0^{\mathrm{y}}} \quad \boldsymbol{e}^{\mathrm{T}} = \tilde{\alpha}\tilde{T}\boldsymbol{i} \quad \tilde{\alpha} = T_0 \alpha. \tag{5.8}$$

Representative values for α and $\tilde{\alpha}$ are also given in table 1.

The flow stress of (3.8) is generalized as follows

$$\left.\begin{aligned} \sigma^{\mathrm{yT}} &= \sigma_0^{\mathrm{yT}} + \sigma_{\mathrm{h}}^{\mathrm{yT}} & \sigma_{\mathrm{h}}^{\mathrm{yT}} &= \sigma_{\mathrm{m}}^{\mathrm{yT}} \tanh \zeta \bar{e}^{\mathrm{p}} \\ \sigma_0^{\mathrm{yT}} &= \sigma_{\mathrm{a}}^{\mathrm{y}} + (\sigma_0^{\mathrm{y}} - \sigma_{\mathrm{a}}^{\mathrm{y}}) e^{-k\tilde{T}} & \sigma_{\mathrm{m}}^{\mathrm{yT}} &= \sigma_{\mathrm{m}}^{\mathrm{y}} e^{-k\tilde{T}} \end{aligned}\right\}, \tag{5.9}$$

where the superscript T is added to emphasize the dependence on temperature. For $\tilde{T}=0$ the form reduces to that used previously with material constants σ_0^{y} and $\sigma_{\mathrm{m}}^{\mathrm{y}}$ continuing to represent the initial yield stress and maximum additional hardening stress, respectively. The rate of decay with temperature has been assumed the same as that for the elastic moduli as simply a matter of convenience. The limiting case of the flow stress for large temperature has been chosen for a specific reason. High temperature is associated with the annihilation of dislocations (annealing), a state at which most metals exhibit no hardening, a feature provided by the assumed form for $\sigma_{\mathrm{m}}^{\mathrm{yT}}$. In addition some metals exhibit a residual flow stress greater than zero but less than the initial flow stress, a feature included here by adding one additional material constant called the athermal stress, $\sigma_{\mathrm{a}}^{\mathrm{y}}$.

Again, dimensionless versions are obtained by using the yield stress, σ_0^{y}, at the reference temperature as a normalizing factor

$$\left.\begin{aligned} \tilde{\sigma}_{\mathrm{a}}^{\mathrm{y}} &= \frac{\sigma_{\mathrm{a}}^{\mathrm{y}}}{\sigma_0^{\mathrm{y}}} \quad \tilde{\sigma}_{\mathrm{m}}^{\mathrm{y}} \equiv \frac{\sigma_{\mathrm{m}}^{\mathrm{y}}}{\sigma_0^{\mathrm{y}}} \\ \tilde{\sigma}^{\mathrm{yT}} &\equiv \frac{\sigma^{\mathrm{yT}}}{\sigma_0^{\mathrm{y}}} = \tilde{\sigma}_0^{\mathrm{yT}} + \tilde{\sigma}_{\mathrm{h}}^{\mathrm{yT}} \\ \tilde{\sigma}_0^{\mathrm{yT}} &\equiv \frac{\sigma_0^{\mathrm{yT}}}{\sigma_0^{\mathrm{y}}} = \tilde{\sigma}_{\mathrm{a}}^{\mathrm{y}} + (1 - \tilde{\sigma}_{\mathrm{a}}^{\mathrm{y}}) e^{-k\tilde{T}} \\ \tilde{\sigma}_{\mathrm{h}}^{\mathrm{yT}} &\equiv \frac{\sigma_{\mathrm{h}}^{\mathrm{yT}}}{\sigma_0^{\mathrm{y}}} = \tilde{\sigma}_{\mathrm{m}}^{\mathrm{y}} e^{-k\tilde{T}} \tanh \zeta \bar{e}^{\mathrm{p}} \end{aligned}\right\}. \tag{5.10}$$

As a generalization of (3.11), we allow C_{d} to depend on temperature with the same exponential decay used previously and assume the dislocation internal energy to be

$$U_\rho = C_{\mathrm{d}}(T)\sqrt{\tilde{\rho}^{\mathrm{d}}} \quad C_{\mathrm{d}}(T) = C_{\mathrm{d}}^{0} e^{-\tilde{k}\tilde{T}}, \tag{5.11}$$

an expression that vanishes with large temperature. Note that we have reverted to the original form for this contribution to the internal energy; the second contribution could be generalized in a similar manner. The material parameter C_{d}^{0} is the value of C_{d} at the reference temperature. The use of (5.11) can be considered an indirect way of representing annealing, rather than reducing dislocation density. A development connecting a reduction of dislocation density with temperature might be worthy of an independent study and is certainly beyond the scope of this paper. The same equations as (5.11) in dimensionless form are

$$\tilde{U}_\rho = \tilde{C}_{\mathrm{d}}^{0} e^{-\tilde{k}\tilde{T}}\sqrt{\tilde{\rho}^{\mathrm{d}}} \quad \tilde{C}_{\mathrm{d}}^{0} = \frac{C_{\mathrm{d}}^{0}}{\sigma_{0}^{\mathrm{y}}}. \tag{5.12}$$

For von Mises plasticity with isotropic hardening, all aspects of the formulation go through as developed previously. In particular, the choice of the evolution function, m^{d}, remains the one given in (3.10) so that the 'effective' dislocation density is the temperature-dependent version of (3.13)

$$\tilde{\tilde{\Sigma}}^{\mathrm{dT}} = \zeta \tilde{C}_{\mathrm{d}}^{0} e^{-\tilde{k}\tilde{T}}[1-\tanh^{2}\zeta\bar{e}^{\mathrm{p}}]. \tag{5.13}$$

The form for β remains similar to (3.15) but is now also dependent on temperature

$$\beta^{\mathrm{T}} = 1-\frac{\tilde{\tilde{\Sigma}}^{\mathrm{dT}}}{\tilde{\sigma}^{\mathrm{yT}}}. \tag{5.14}$$

If $\tilde{T}=0$ when $\bar{e}^{\mathrm{p}}=0$ then the beginning value for β^{T} becomes $\beta_0 = 1-\zeta\tilde{C}_{\mathrm{d}}^{0}$, similar to the expression for the uncoupled theory. The plastic power and the dissipation power are also similar in form to (2.18) and depend on temperature

$$\tilde{P}^{\mathrm{PT}} = \dot{\lambda}\tilde{\sigma}^{\mathrm{yT}} \quad \tilde{D}^{\mathrm{PT}} = \beta^{\mathrm{T}}\tilde{P}^{\mathrm{PT}}. \tag{5.15}$$

Since the two expressions in (5.15) do not represent total derivatives it is now incorrect to refer to these quantities as dissipation rate and rate of plastic work.

The thermally-dependent dimensionless form for the first law given in (4.12) becomes

$$\dot{\tilde{T}} = \beta^{\mathrm{T}} r_{\mathrm{c}}^{\mathrm{y}} \dot{\bar{e}}^{\mathrm{p}} \tilde{\sigma}^{\mathrm{yT}}. \tag{5.16}$$

The result is that a nonlinear ordinary differential equation must be solved for T. The uncoupled case is recovered by setting $\tilde{k}=0$.

(*b*) *Representative results*

To illustrate the effects of coupling, we again provide plots of flow stress and temperature as functions of effective plastic strain. However, instead of just ζ and $r_{\mathrm{c}}^{\mathrm{y}}$ we have the additional parameters of $\tilde{k}$ and $\tilde{\sigma}_{\mathrm{a}}^{\mathrm{y}}$ whose effects must be exhibited.

In figures 6 and 7, we show the effect of the decay parameter, $\tilde{k}$, on the flow stress and the temperature, respectively, for $\tilde{\sigma}_{\mathrm{m}}^{\mathrm{y}}=0.5$, $\zeta=16$ and $\tilde{\sigma}_{\mathrm{a}}^{\mathrm{y}}=1$. Even though the flow stress always hardens, the effect of temperature coupling is to provide enhanced softening with increasing values of $\tilde{k}$. There is a slightly smaller

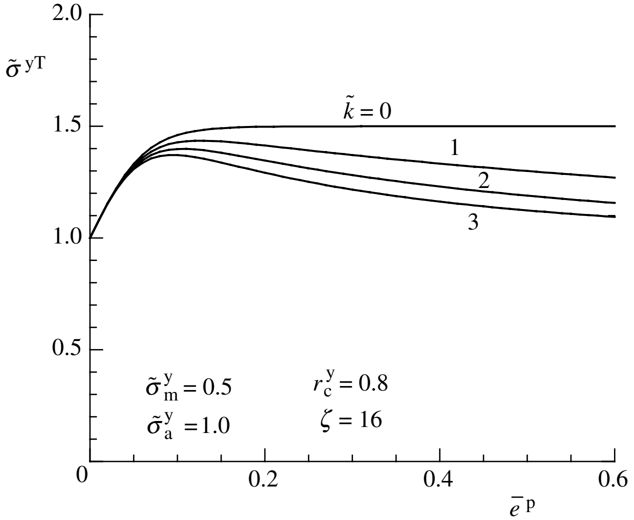

Figure 6. Flow stress versus effective plastic strain for the coupled theory when material properties depend on temperature.

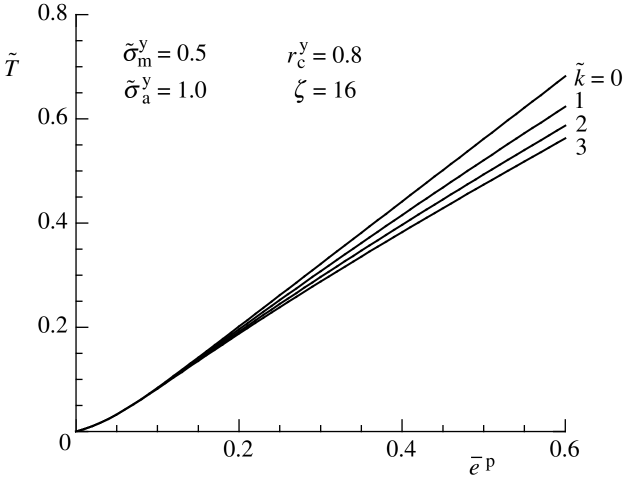

Figure 7. Temperature versus effective plastic strain for the coupled theory when material properties depend on temperature.

increase in temperature for a given amount of plastic strain when the thermal effects on material parameters is taken into account. Surprisingly, there is practically no change of β with $\tilde{k}$ so a plot is not shown.

If $\tilde{\sigma}^{y}_{a} < 1$ then the flow stress may decay below the initial yield stress as shown in figure 8.

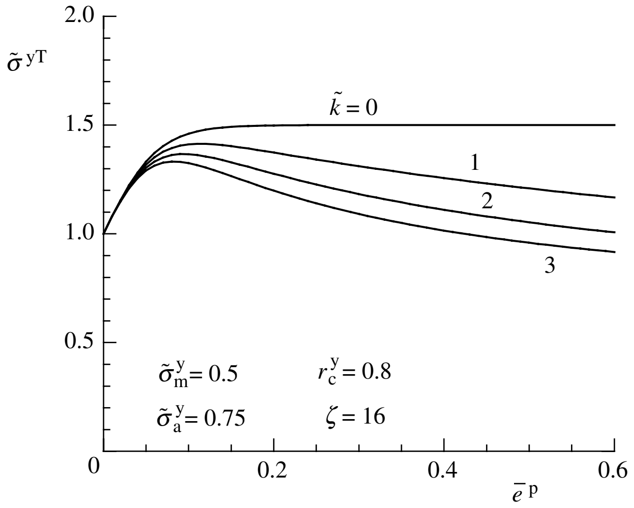

Figure 8. Flow stress versus effective plastic strain for the coupled theory and a reduced value for the athermal stress when material properties depend on temperature.

(*c*) *Additional temperature*

Recall that we have assumed the additional temperature, T_{a}, to be much smaller than the reference temperature. We are now in a position to obtain an estimate for T_{a} and check the assumption.

In order to evaluate the relative significance of the terms in the equation, we make the restriction of uniaxial stress so that the only non-zero stress, $\boldsymbol{\sigma}$, is related to the elastic strain, e^{e}, through Young's modulus, E. With uniaxial stress, and the assumed form of Mises plasticity, the axial plastic strain, e^{p}, is identical to the effective plastic strain. Therefore, the elastic internal energy and constitutive relation reduce to

$$U_{\mathrm{e}} = \tfrac{1}{2}E(e^{\mathrm{e}})^2 \quad \sigma = Ee^{\mathrm{e}} \quad e^{\mathrm{e}} = e - \bar{e}^{\mathrm{p}} - e^{\mathrm{T}}. \tag{5.17}$$

The thermal dependence of Young's modulus and the thermal strain are

$$\tilde{E} = \tilde{E}_0 e^{-k\tilde{T}} \quad e^{\mathrm{T}} = \tilde{\alpha}\tilde{T}, \tag{5.18}$$

in which $\tilde{E}$ is the normalized form of Young's modulus, and $\tilde{E}_0$ is its value at the reference temperature, T_0. Typical values for E_0 are given in table 1. We separate the contributions to the additional temperature into three parts according to the individual sources

$$T_{\mathrm{a}} = T_{\mathrm{a}}^{\mathrm{e}} + T_{\mathrm{a}}^{\alpha} + T_{\mathrm{a}}^{\rho}, \tag{5.19}$$

where T_{a}^{α} denotes the contribution from the thermal strain, T_{a}^{E} the part from Young's modulus, and T_{a}^{ρ} from the dislocation internal energy. With the use of (5.4), these contributions are

$$T_{\mathrm{a}}^{\alpha} = \frac{T}{c_{\mathrm{v}}}\frac{\partial U_{\mathrm{e}}}{\partial e^{\mathrm{T}}}\frac{\partial e^{\mathrm{T}}}{\partial T} \quad T_{\mathrm{a}}^{\mathrm{E}} = \frac{T}{c_{\mathrm{v}}}\frac{\partial U_{\mathrm{e}}}{\partial E}\frac{\partial E}{\partial T} \quad T_{\mathrm{a}}^{\rho} = \frac{T}{c_{\mathrm{v}}}\frac{\partial U_{\rho}}{\partial C_{\rho}}\frac{\partial C_{\rho}}{\partial T}. \tag{5.20}$$

In dimensionless form these equations become

$$\left.\begin{aligned} \tilde{T}_{\mathrm{a}}^{\alpha} &= -(1+\tilde{T})r_{\mathrm{c}}^{\mathrm{y}}\tilde{\alpha}\tilde{\sigma} \qquad & \tilde{\sigma} &= \frac{\sigma}{\sigma_0^{\mathrm{y}}} \\ \tilde{T}_{\mathrm{a}}^{E} &= -(1+\tilde{T})r_{\mathrm{c}}^{\mathrm{y}}\tilde{k}\tilde{U}_{\mathrm{e}} \qquad & \tilde{U}_{\mathrm{e}} &= \frac{U_{\mathrm{e}}}{\sigma_0^{\mathrm{y}}} \\ \tilde{T}_{\mathrm{a}}^{\rho} &= -(1+\tilde{T})r_{\mathrm{c}}^{\mathrm{y}}\tilde{k}\tilde{U}_{\rho} \qquad & \tilde{U}_{\rho} &= \frac{U_{\rho}}{\sigma_0^{\mathrm{y}}} \end{aligned}\right\}. \tag{5.21}$$

Next, we provide an order-of-magnitude analysis for these possible additions to temperature. If $\sigma_{\mathrm{y}}^{\mathrm{m}}=0.5$ then the upper bound to the yield stress and the stress is $\tilde{\sigma}^{\mathrm{UB}}=1.5$. Assume $r_{\mathrm{c}}^{\mathrm{y}}=1$, which appears to be an upper bound based on table 1. Suppose further that the total strain is $e=0.5$, which is approximately equal to the total plastic strain. Since $\beta^{\mathrm{T}}\le 1$ the entropy temperature obtained from (5.16) is bounded by

$$\tilde{T} \le \bar{e}\tilde{\sigma} = (0.5)(1.5) = 0.75. \tag{5.22}$$

The dimensionless coefficients of thermal expansion in table 1 are all less than 10^{-2}. Therefore an estimate of the maximum value of additional temperature due to the thermal strain is

$$\tilde{T}_a^{\alpha} = -r_c^y\tilde{\alpha}\tilde{\sigma}(1+\tilde{T}) \approx -(1)(10^{-2})(1.5)(1.75) \approx -0.03, \tag{5.23}$$

an insignificant contribution.

To estimate the contribution due to the thermal dependence of Young's modulus on temperature, suppose $\tilde{k}=0.5$. Then, for the dimensionless temperature of (5.22) it follows that $e^{-\tilde{k}\tilde{T}}=0.7$. From table 1, an upper bound for $\tilde{E}_0$ is 10^3. An upper bound on the elastic strain is the maximum stress divided by Young's modulus or

$$e^{\mathrm{e}} = \tilde{\sigma}_{\max}/\tilde{E} = (1+\tilde{\sigma}_{\mathrm{m}}^{\mathrm{y}})/(e^{-\tilde{k}\tilde{T}}\tilde{E}_0) = 1.5/(0.7\times 10^3) = 2\times 10^{-3}. \tag{5.24}$$

Then

$$\begin{aligned} \tilde{T}_{\mathrm{a}}^{\mathrm{e}} &= -\tfrac{1}{2}r_{\mathrm{c}}^{\mathrm{y}}\tilde{k}\tilde{E}_0 e^{-\tilde{k}\tilde{T}}(e^e)^2(1+\tilde{T}) \approx -\tfrac{1}{2}(1.0)(0.5)(10^3)(0.7)(2\times 10^{-3})(1.75) \\ &= -0.001, \end{aligned} \tag{5.25}$$

a contribution even smaller than that due to the coefficient of thermal expansion.

Finally, we look at the potential additional temperature due to plastic strain. From (5.12) the dimensionless dislocation internal energy is

$$\tilde{U}_{\rho} = \tilde{C}_{\mathrm{d}}^{0}e^{-\tilde{k}\tilde{T}}\tanh\zeta\bar{e}^{\mathrm{p}}. \tag{5.26}$$

If $\zeta=16$ and $\bar{e}^{\mathrm{p}}=0.5$, then it follows that $\tanh\zeta\bar{e}^{p}=1$. If we choose $\beta_0=0.5$ and since $\beta_0=1-\zeta\tilde{C}_{\mathrm{d}}^{0}$, we obtain $\tilde{C}_{\mathrm{d}}^{0}=0.03$ and

$$\tilde{U}_{\rho} \approx (0.03)(0.7)(1) = 0.02. \tag{5.27}$$

The predicted increase in temperature involving this term is

$$\tilde{T}_{\mathrm{a}}^{\rho} = -r_{\mathrm{c}}^{\mathrm{y}}\tilde{k}\tilde{U}_{\rho}(1+\tilde{T}) \approx -(1.0)(0.5)(0.02)(1.75) = -0.02. \tag{5.28}$$

At least for the chosen values of material parameters, this contribution is also relatively insignificant and the assumption of (5.3) that $T \approx T_\eta$ indeed holds. One side observation is that all three of the potential contributions are negative, indicating that the predicted temperature excluding these terms will be on the high side.

It should also be noted that the increment in temperature is negative for initial elastic loading and positive for initial elastic unloading in agreement with the experimental measurements of Pieczyska (1999). Here we have purposely emphasized the regime of plastic deformations and considered the thermoelastic effect to be neglible.

6. Summary

This paper provides a thermodynamically consistent formulation of conventional plasticity constitutive equations with thermal effects included but under the assumptions that heat conduction and external heat sources are negligible. The formulation is motivated by the desire to emphasize the point made by Rosakis *et al.* (2000) that β is not a constant, but rather, a function that follows from a formulation that includes an additional term in the internal energy. The formulation does not include many important factors such as a rigorous treatment of large deformations, a general coupled form of the internal energy, and thermal dependence of all material parameters.

Essential aspects of the development are the following:

(i) The formulation is based on the premise that, in addition to the usual entropy and elastic contributions to internal energy, there should be a contribution that depends on dislocation density.
(ii) With a suitable interpretation of internal variables, the formulation can utilise equivalently either dislocation density or effective plastic strain.
(iii) The use of dislocation density as a primary variable allows the representation for a partially-annealed metal through the choice of a non-zero value for initial dislocation density.
(iv) The existence of the dislocation internal energy indicates that not all of the plastic power contributes to dissipation and provides an explanation for why a reduction factor (beta) is necessary in many elementary formulations to obtain agreement with observed increases in temperature.
(v) Two elementary terms for the dislocation internal density provide the capability for obtaining a wide range in forms for the function β.
(vi) The uncoupled formulation provides a very simple expression for dissipation and, consequently, for increase in temperature.
(vii) The formulation is sufficiently simple so that it can be easily altered to match experimental data giving β as a function of effective plastic strain.
(viii) For metals that exhibit a significant flow stress-dependence on strain rate, there is a corresponding enhancement in the expression for dissipation.
(ix) An assumption that the additional temperature is insignificant rendered a coupled theory almost as simple as the uncoupled form. An *a posterior*

argument indicates that the assumption may be plausible for many problems.

(x) The coupled theory, not surprisingly, indicates that temperature dependence provides a modified strain-hardening form of plasticity. For certain values of material constants, strain softening is exhibited with the resulting possibility of loss of ellipticity and ill-posedness (Neilsen & Schreyer 1993).

In summary, a mathematical structure has been provided for a thermodynamical treatment of plasticity that is theoretically rigorous, that is a natural evolution from a widely-used engineering model of plasticity, and that offers a suitable framework for describing the β-effect. The simplicity of the formulation provides a basis for those new to the subject to understand and appreciate the efforts of those who have made prior contributions to the field.

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